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1992 J. Phys. A: Math. Gen. 25 4699

(http://iopscience.iop.org/0305-4470/25/17/026)

J. Phys. A: Math. Gen. **25** (1992) 4699–4704. Printed in the UK

# Comment on generalized parasupersymmetric quantum mechanics

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Received 3 January 1991, in final form 13 April 1992

**Abstract.** We construct the arbitrary order parasupersymmetric quantum mechanics of one boson and one parafermion degrees of freedom. The parasupersymmetry algebra is $Q_j^{2j}Q_j^\dagger + Q_j^{2j-1}Q_j^\dagger Q_j + \ldots + Q_j^\dagger Q_j^{2j} = \alpha_j Q^{2j-1}H$ ($\alpha_j = \frac{2}{3}j(j+1)(2j+1)$), $Q_j^{2j+1} = 0$, $[Q_j, H] = 0$, where $2j$ represents the order of parasupersymmetry, $H$ is the Hamiltonian and $Q_j$ is the parasupercharge.

Recently the interest of physicists has turned to the possibilities of statistically exotic behaviour of particles. Its scope is expected to spread over a wide range of physics. For example, the anyon [1] is one such case. Here we consider parastatistics [2], applying it to quantum mechanics associated with supersymmetry [3].

In the case of ordinary $N = 1$ supersymmetric quantum mechanics, the supercharge $Q$ and the Hamiltonian $H$ make the supersymmetric algebra [3]

$$\{Q, Q^\dagger\} = QQ^\dagger + Q^\dagger Q = H \qquad Q^2 = 0 \qquad [H, Q] = 0. \tag{1}$$

We shall try to generalize the parasupersymmetric quantum mechanics which was introduced by Rubakov and Spiridonov [4]. They constructed the second-order parasupersymmetric quantum mechanics. We will generalize it to higher order. In generalized parasupersymmetric quantum mechanics, fermions obey parastatistics; the same kind of fermion can occupy the same state $n$ times ($n = 1$, ordinary fermion; $n = 2$, Rubakov's case).

We shall begin by studying the parasupercharge which consists of the direct product of the parafermionic and bosonic operators. In the case of second-order parasupersymmetric quantum mechanics, the supercharge $Q$ and the Hamiltonian $H$ obey the parasupersymmetric algebra

$$Q^2Q^\dagger + QQ^\dagger Q + Q^\dagger Q^2 = 4QH \qquad Q^3 = 0 \qquad [H, Q] = 0. \tag{2}$$

Note that this time the square of $Q$ does not vanish, but the cube of $Q$ does. The Hamiltonian does not have a direct representation by $Q$ and $Q^\dagger$. Nevertheless, the fermionic creation (annihilation) operator $f^\dagger$ ($f$) and the fermionic part of the Hamiltonian $H_F$ still preserve the algebra

$$[f, H_F] = f \qquad [f^\dagger, H_F] = -f^\dagger \qquad H_F = \tfrac{1}{2}[f^\dagger, f]. \tag{3}$$

Therefore $f$, $f^\dagger$ and $2H_F$ perform the algebras of sl(2; $\mathbb{C}$). Its fundamental representation corresponds to an ordinary fermion. Higher-dimensional representations correspond to higher-order parastatistics respectively. The construction of the higher representation of sl(2; ℂ) is already familiar among physicists as the composition of angular momentum. Mathematically the representation for the angular momentum $j$ coincides with the parafermionic operators of order $n=2j$. The identification $n=2j$ will help us to clarify the algebraic structure of the parafermionic operators. Let us redefine the creation (annihilation) operator of the parafermion of the order $n=2j$ as $f_j^\dagger$ ($f_j$) and the supercharge as $Q_j$. Subscript $j$ is to distinguish the dimension of representation (the dimension is $2j+1$) and of course it is mathematically related to angular momentum $j$ described above.

In the following we shall prove that operators $f_j^\dagger$ ($f_j$) obey the identity

$$f_j^{2j}f_j^\dagger+f_j^{2j-1}f_j^\dagger f_j+f_j^{2j-2}f_j^\dagger f_j^2+\ldots+f_j^\dagger f_j^{2j}=\alpha_j f_j^{2j-1} \tag{A}$$

where

$$\alpha_j=\tfrac{2}{3}j(j+1)(2j+1) \qquad (j=\tfrac{1}{2},1,\tfrac{3}{2},2,\ldots).$$

In order to prove (A) we choose the Hermitian representation of $f_j$ ($f_j^\dagger$).

$$\begin{aligned}\langle j,m+1|f_j^\dagger|j,m\rangle&=\sqrt{(j-m)(j+m+1)} \qquad (m=-j,-j+1,\ldots,j-1)\\ \langle j,m-1|f_j|j,m\rangle&=\sqrt{(j+m)(j-m+1)} \qquad (m=-j+1,\ldots,j-1,j)\end{aligned} \tag{4}$$

otherwise

$$\langle j,k|f_j^{(\dagger)}|j,l\rangle=0.$$

Here the state $|j,m\rangle$ has been defined as

$$\tfrac{1}{2}[f_j^\dagger,f_j]|j,m\rangle=m|j,m\rangle \qquad -j\leqslant m\leqslant j$$

and this state represents $(m+j)$-parafermions. We calculate term by term:

$$\begin{aligned}
&(f_j^{2j}f_j^\dagger|j,j\rangle=0)\\
&\langle j,-j|f_j^{2j}f_j^\dagger|j,j-1\rangle=2j\prod_{m=-j+1}^{j-1}\sqrt{(j+m)(j-m+1)}\\
&\langle j,-j+1|f_j^{2j-1}f_j^\dagger f_j|j,j\rangle=2j_m\prod_{m=-j+2}^{j}\sqrt{(j+m)(j-m+1)}\\
&\langle j,-j|f_j^{2j-1}f_j^\dagger f_j|j,j-1\rangle=2(2j-1)\prod_{m=-j+1}^{j-1}\sqrt{(j+m)(j-m+1)}\\
&\langle j,-j+1|f_j^{2j-2}f_j^\dagger f_j^2|j,j\rangle=2(2j-1)\prod_{m=-j+2}^{j}\sqrt{(j+m)(j-m+1)}\\
&\vdots\\
&\langle j,-j|f_j^{2j-k}f_j^\dagger f_j^k|j,j-1\rangle=(2j-k)(k+1)\prod_{m=-j+1}^{j-1}\sqrt{(j+m)(j-m+1)}\\
&\langle j,-j+1|f_j^{2j-k-1}f_j^\dagger f_j^{k+1}|j,j\rangle=(2j-k)(k+1)\prod_{m=-j+2}^{j}\sqrt{(j+m)(j-m+1)}\\
&\vdots\\
&\langle j,-j|f_jf_j^\dagger f_j^{2j-1}|j,j-1\rangle=2j\prod_{m=-j+1}^{j-1}\sqrt{(j+m)(j-m+1)}\\
&\langle j,-j+1|f_j^\dagger f_j^{2j}|j,j\rangle=2j\prod_{m=-j+2}^{j}\sqrt{(j+m)(j-m+1)}\\
&(f_j^\dagger f_j^{2j}|j,j-1\rangle=0).
\end{aligned} \tag{5}$$

Summing up all terms, we obtain

$$\begin{aligned}
&\langle j, -j+1|(f_j^{2j}f_j^\dagger + f_j^{2j-1}f_j^\dagger f_j + f_j^{2j-2}f_j^\dagger f_j^2 + \ldots + f_j^\dagger f_j^{2j})|j, j\rangle \\
&\qquad = \tfrac{2}{3}j(j+1)(2j+1) \prod_{m=-j+2}^{j} \sqrt{(j+m)(j-m+1)} \\
&\qquad = \alpha_j \langle j, -j+1|f_j^{2j-1}|j, j\rangle \\
&\langle j, -j|(f_j^{2j}f_j^\dagger + f_j^{2j-1}f_j^\dagger f_j + f_j^{2j-2}f_j^\dagger f_j^2 + \ldots + f_j^\dagger f_j^{2j})|j, j-1\rangle \\
&\qquad = \alpha_j \langle j, -j|f_j^{2j-1}|j, j-1\rangle
\end{aligned} \tag{6}$$

where we have used

$$\sum_{k=0}^{2j-1} (2j-k)(k+1) = \tfrac{2}{3}j(j+1)(2j+1) = \alpha_j.$$

All the other matrix elements vanish. Therefore we have the operator's identity (A).

Next we can derive the identity

$$Q_j^{2j}Q_j^\dagger + Q_j^{2j-1}Q_j^\dagger Q_j + Q_j^{2j-2}Q_j^\dagger Q_j^2 + \ldots + Q_j^\dagger Q_j^{2j} = \alpha_j Q_j^{2j-1} H \tag{B}$$

where

$$Q_j = a^\dagger f_j \qquad Q_j^\dagger = a f_j^\dagger \qquad H = \tfrac{1}{2}\{a^\dagger, a\} + \tfrac{1}{2}[f_j^\dagger, f_j]$$

between the Hamiltonian $H$ and the supercharge $Q_j$. The formula (A) plays a fundamental role here. The left-hand side of (B) becomes

$$\begin{aligned}
&a^{\dagger 2j} a f_j^{2j} f_j^\dagger + a^{\dagger 2j-1} a a^\dagger f_j^{2j-1} f_j^\dagger f_j + \ldots + a a^{\dagger 2j} f_j^\dagger f_j^{2j} \\
&\qquad = a^{\dagger 2j-1} H_{\mathrm{B}} (f_j^{2j} f_j^\dagger + f_j^{2j-1} f_j^\dagger f_j + \ldots + f_j^\dagger f_j^{2j}) + a^{\dagger 2j-1} \tfrac{1}{2}\{-f_j^{2j} f_j^\dagger + f_j^{2j-1} f_j^\dagger f_j \\
&\qquad\quad + \ldots + 2(k-\tfrac{1}{2}) f_j^{2j-k} f_j^\dagger f_j^k + \ldots + 2(2j-\tfrac{1}{2}) f^\dagger f^{2j} S \\
&\qquad = a^{\dagger 2j-1} H_{\mathrm{B}} \alpha_j f_j^{2j-1} + a^{\dagger 2j-1} \cdot \tfrac{1}{2} \cdot \tfrac{4}{3} \cdot j(j+1)(2j+1) f_j^{2j-1} H_{\mathrm{F}} \\
&\qquad = \alpha_j (a^{\dagger 2j-1} f_j^{2j-1})(H_{\mathrm{B}} + H_{\mathrm{F}}) \\
&\qquad = \alpha_j Q^{2j-1} H
\end{aligned} \tag{7}$$

where

$$H_{\mathrm{B}} = \tfrac{1}{2}\{a^\dagger, a\} \qquad H_{\mathrm{F}} = \tfrac{1}{2}[f_j^\dagger, f_j].$$

Finally we reach the superalgebra

$$\begin{aligned}
&Q_j^{2j}Q_j^\dagger + Q_j^{2j-1}Q_j^\dagger Q_j + Q_j^{2j-2}Q_j^\dagger Q_j^2 + \ldots + Q_j^\dagger Q_j^{2j} = \alpha_j Q_j^{2j-1} H \\
&Q_j^{2j+1} = 0 \qquad [Q_j, H] = 0.
\end{aligned} \tag{C}$$

The bosonic part of the Hamiltonian $H_{\mathrm{B}}$ has positive definiteness, since $H_{\mathrm{B}}$ is the Hamiltonian of an ordinary harmonic oscillator. Its lowest energy is $\frac{1}{2}$. On the other hand, the parafermionic part of the Hamiltonian $H_{\mathrm{F}}$ of order $2j$ has the series of the eigenvalues: $-j, -j+1, \ldots, j$. In the case of $j=\frac{1}{2}$ (ordinary SUSY), the lowest energy of the total Hamiltonian $H = H_{\mathrm{B}} + H_{\mathrm{F}}$ is zero, and $H$ has positive semidefiniteness. In the case of $j > \frac{1}{2}$, unfortunately we cannot require that the spectrum of $H$ is non-negative automatically without some other physical requirement. The states are $2j$-fold degenerate, but there are a few exceptions. For example, we may assume that there exists the unique ground state with the energy $E_0$. Energy levels with $E \gg E_0$ are $2j$-fold degenerate. More precisely, the ground state is non-degenerate, the first excited state is doubly degenerate, the second excited state is triply degenerate, ..., $i$th ($i<2j$) excited state is $(i+1)$-fold degenerate. In the case $i \geqslant 2j$, states are $2j$-folded. Details of the spectrum will be discussed elsewhere [5].

Rubakov and Spiridonov [4] treat a more general case that the parasupercharge cannot be split into the direct product of the parafermionic and bosonic operators. Their Hamiltonian can possess the positive semidefiniteness of the energy under some physical conditions. We will generalize their parasupersymmetric quantum mechanics to higher order. Basically they represent the parafermion of order $2j$ in a $(2j+1)\times(2j+1)$-matrix. Thus the parasupercharge is represented in a more complicated matrix

$$(Q)_{i+1,i} = \gamma_i a_i^\dagger \qquad (Q^\dagger)_{i,i+1} = \gamma_i a_i \qquad (i=1,2,\ldots,2j) \tag{8}$$

where $a_i^{(\dagger)}$ $(i=1,2,\ldots,2j)$ represent $2j$ bosonic annihilation (creation) operators. Here the states are represented in $(2j+1)$-column vectors. The state, which has only the $i$th component from the bottom, has parafermionic number $(i-1)$. We can also write the charge in terms of $f_j$ and $f_j^\dagger$ only:

$$Q = \frac{f_j f_j^\dagger}{|\gamma_1\gamma_2\cdots\gamma_{2j}|^2}\left(a_1^\dagger f_j^{\dagger 2j-1} f_j^{2j} + a_2^\dagger f_j^{\dagger 2j-2} f_j^{2j} f_j^\dagger + a_3^\dagger f_j^{\dagger 2j-3} f_j^{2j} f_j^{\dagger 2} + \ldots + a_{2j}^\dagger f_j^{2j} f_j^{\dagger 2j-1}\right). \tag{9}$$

The constants $\gamma_i$ $(i=1,2,\ldots,2j)$ are now defined as

$$\gamma_{2j-k+1} = \langle k-1|f_j|k\rangle$$

where $|k\rangle$ $(k=1,\ldots,2j)$ denotes the normalized state vector with the parafermionic number $k$. The Hamiltonian is expressed in the matrix

$$H = \frac{1}{2\alpha_j}\sum_{i=1}^{2j}\gamma_i^2\{a_i^\dagger, a_i\} + \frac{1}{2\alpha_j}\operatorname{diag}(g_1, g_2, \ldots, g_{2j+1}) \tag{10}$$

where

$$g_1 = \left(\gamma_1^2 + 2\sum_{i=2}^{2j}\gamma_i^2\right)[a_1, a_1^\dagger] + \left(\gamma_2^2 + 2\sum_{i=3}^{2j}\gamma_i^2\right)[a_2, a_2^\dagger] + \ldots + \gamma_{2j}^2[a_{2j}, a_{2j}^\dagger]$$

$$\vdots$$

$$g_{k+1} = g_k - 2\alpha_j[a_k, a_k^\dagger]$$

$$\vdots$$

$$\begin{aligned} g_{2j+1} &= -\gamma_1^2[a_1, a_1^\dagger] - (2\gamma_1^2 + \gamma_2^2)[a_2, a_2^\dagger] - \ldots - \left(2\sum_{i=1}^{2j-1}\gamma_i^2 + \gamma_{2j}^2\right)[a_{2j}, a_{2j}^\dagger] \\ &= g_{2j} - 2\alpha_j[a_{2j}, a_{2j}^\dagger] \end{aligned}$$

and

$$\gamma_i = \sqrt{i(2j-i+1)}.$$

The constants $\gamma_i$ are based upon the higher representation sl(2; $\mathbb{C}$) (cf equation (4)). After straightforward calculation we can prove that the superalgebra (C) is still valid, provided that

$$a_{i+1}a_{i+1}^\dagger = a_i^\dagger a_i + c_i \qquad c_i = \text{constant} \qquad (i=1,\ldots,2j-1). \tag{11}$$

A proof will be given in a future publication [5].

For example, assuming $\forall c_i = 0$ we would get simple solutions

$$a_{i+1} = \pm a_i^{\dagger} \qquad (i = 1, \dots, 2j-1) \qquad \text{and} \qquad a_1 = a. \tag{12}$$

The solutions (12) are preferable for the positive semidefiniteness of the energy. It was demonstrated in [6] in the case of the order $n = 2$.

Parasupersymmetric quantum mechanics was also discussed in terms of the Green-Cusson ansätze [7] by Beckers and Debergh [8]. By the Green ansätze, the parafermionic annihilation operator of the order $n = 2j$ is written as

$$F_j = \sum_{i=1}^{2j} \xi_i \tag{13}$$

where the Green components $\xi_i$ obey the relations

$$\begin{aligned} &\{\xi_i, \xi_i^{\dagger}\} = 1 \\ &\{\xi_i, \xi_i\} = \{\xi_i^{\dagger}, \xi_i^{\dagger}\} = 0 \\ &[\xi_i, \xi_j] = [\xi_i, \xi_j^{\dagger}] = [\xi_i^{\dagger}, \xi_j^{\dagger}] = 0 \qquad (i \neq j). \end{aligned} \tag{14}$$

The Green components are similar to the ordinary fermions, though $\xi_i$ and $\xi_j^{(\dagger)}$ $(i \neq j)$ commute each other in (14), while the ordinary fermionic operators anti-commute. In Cusson's realization of the Green ansätze the Green component can also be represented by the direct product of the ordinary fermionic annihilation operators and the Dirac matrices [7]. The Dirac operators need to transfer the anti-commutability to the commutability. This representation is completely equivalent to the following direct product representation:

$$\begin{aligned} F_j &= \theta \otimes \underbrace{I \otimes \dots \otimes I}_{(2j-1)I's} + I \otimes \theta \otimes \underbrace{I \otimes \dots \otimes I}_{(2j-2)I's} + \dots + \underbrace{I \otimes \dots \otimes I}_{(2j-1)I's} \otimes \theta \\ &= \sum_{i=1}^{2j} \underbrace{I \otimes \dots \otimes I}_{(i-1)I's} \otimes \theta \otimes \underbrace{I \otimes \dots \otimes I}_{(2j-i)I's} \end{aligned} \tag{15}$$

where

$$\theta = f_{1/2} \qquad \text{and} \qquad \{\theta, \theta^{\dagger}\} = I \qquad \{\theta, \theta\} = 0.$$

If we put

$$f^i = \underbrace{I \otimes \dots \otimes I}_{(i-1)I's} \otimes \theta \otimes \underbrace{I \otimes \dots \otimes I}_{(2j-1)I's}$$

the parafermionic annihilation operator is written as

$$F_j = \sum_{i=1}^{2j} f^i.$$

This time $f^i$ can be identified with the Green components because

$$\{f^i, f^{i\dagger}\} = \underbrace{I \otimes \ldots \otimes I}_{pI's}$$

$$\{f^i, f^i\} = \{f^{i\dagger}, f^{i\dagger}\} = 0 \tag{16}$$

$$[f^i, f^j] = [f^i, f^{j\dagger}] = [f^{i\dagger}, f^{j\dagger}] = 0 \qquad (i \neq j).$$

Although our parafermionic annihilation (creation) operator and Green–Cusson annihilation (creation) operator are constructed in a different way, the algebraic structure is equivalent. Representing $F_j$ for the operator $f_j$, the identity (A) is still kept valid. Therefore, both the definitions of the parasupercharge $Q_j = a^\dagger F_j$ and the supercharge (9) also lead to the identity (B) and the superalgebra (C).

## Acknowledgments

The author is very grateful to all the members of the Department of Applied Physics, Seikei University and would also like to thank Professor Naotaka Yoshinaga for valuable discussions.

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